HW. # 18

Homework problems are taken from several textbooks. The problems are color coded to indicate level of difficulty. The color green indicates an elementary problem, which you should be able to solve effortlessly. Yellow means that the problem is somewhat harder. Red indicates that the problem is hard. You should attempt the hard problems especially.

Find the maximum and minimum values of the function subject to the given constraint. Verify in each case that the constraint equation has a closed and bounded graph.

1. $f(x, y) = 8x + 3y$; $(x-1)^2 + (y+2)^2$.
2. $f(x, y) = xy$; $\dfrac{x^2}{25} + \dfrac{y^2}{4} = 1$.
3. $f(x, y) = 8x + 27y$; $x^4 + y^4 = 1$
4. $f(x, y) = x^2 + y^3$; $x^2 + y^2 = 49$
5. $f(x, y, z) = xyz$; $x^2 + y^2 + z^2 = a^2$
6. $f(x, y, z) = z - x^2 - y^2$; $\dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} = 1$
7. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$

Find the location of possible local extrema, subject to the given constraint. Use the second derivative test if possible to determine whether the function has a local maximum or minimum.

8. $f(x, y) = x^2 + y^2$; $5x - 9y = 1$
9. $f(x, y) = -x^2 - y^2$; $x = y^2$
10. $f(x, y) = (x^3 - x)y^2$; $2x + 3y = 0$
11. $f(x, y, z) = 2x^2 + y^2 + 4z^2$; $x + y + z = 1$
12. $f(x, y, z) = x + y + z$; $z = x^2 + y^2$

Find the maximum and minimum values of the given function with the given constraints.

13. $f(x, y, z) = 3x^2 + y^2 + 3z^2$; $x^2 + y^2 + z^2 = 1$, $x - y + 5z = 0$

14. $f(x, y, z) = x + y + z$; $x^2 + y^2 + z^2 = 1$, $x + 2y + z = 0$

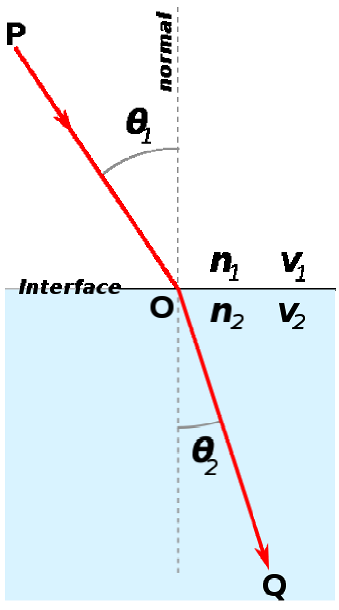
Solve the following applied problems.

15. A company incurs costs of $C = 5x^2 + 2xy + 3y^2 + 800$ (in thousands of dollars) when it produces x thousand units of one product and y thousand units of another. Its production capacity is such that $x + y = 39$. At what production levels will the company's costs be minimized? What will be the corresponding total cost?

16. A principle of optics is that light traveling through different media (where its velocity will vary) will follow a path that minimizes the time for its travel. This principle can be used to derive **Snell's law of refraction.** Suppose that light has velocity v_1 in medium 1 and velocity v_2 in medium 2. If a light ray incident on the interface between medium 1 and medium 2 forms an angle of θ_1 with the normal to the interface, then it will be **refracted** (bent) so that it travels along a line in medium 2 that forms an angle of θ_2 with the normal, where θ_2 satisfies

$$\frac{Sin\theta_1}{Sin\theta_2} = \frac{v_1}{v_2}$$

Use the minimum-time principle and Lagrange multipliers to derive Snell's law.



17. Lagrange multipliers are often used in inverstment strategies. Suppose that an investor has a constant **risk aversion factor** a (larger a means greater fear of stock prices dropping over time) that can be determined empirically. Suppose further that this investor is interested in four specific stocks whose average daily rates of return (over a span of a month, say) are r_1, r_2, r_3, and r_4, and whose variances over the same period are s_1^2, s_2^2, s_3^2, and s_4^2. (Variance is essentially the square of the standard deviation.) These values persumably give some estimate of the stock price behavior into the future. It can be shown that if w_1, w_2, w_3, and w_4 are the fractions of the investor's money that she allocates to buy the respective stocks, then she shood choose them so as to maximize

$$f(w_1,w_2,w_3,w_4) = r_1w_1 + r_2w_2 + r_3w_3 + r_4w_4 - a(s_1^2w_1^2 + s_2^2w_2^2 + s_3^2w_3^2 + s_4^2w_4^2),$$

subject to the constraint $w_1 + w_2 + w_3 + w_4 = 1$.

(a) Show that f is maximized when

$$w_1 = \frac{1 - \dfrac{r_2 - r_1}{2as_2^2} - \dfrac{r_3 - r_1}{2as_3^2} - \dfrac{r_4 - r_1}{2as_4^2}}{1 + \dfrac{s_1^2}{s_2^2} + \dfrac{s_1^2}{s_3^2} + \dfrac{s_1^2}{s_4^2}}$$

$$w_2 = \frac{(r_2 - r_1) + 2as_1^2w_1}{2as_2^2}$$

$$w_3 = \frac{(r_3 - r_1) + 2as_1^2w_1}{2as_3^2}$$

$$w_4 = \frac{(r_4 - r_1) + 2as_1^2w_1}{2as_4^2}$$

(b) Risk aversion can be estimated in the following way: If you are indifferent between two portfolios, one of which has modest average rate of return r and relatively small variance s^2 and the second of which has a higher average rate of return R and accompanying larger variance S^2, then your risk aversion factor is

$$a = \frac{R - r}{S^2 - s^2}$$

Determine the risk aversion factor for an investor who is indifferent between one portfolio with r = 5% and $s^2 = 1\%$ and another with R = 8% and $S^2 = 4\%$.